On the drag of a flat plate at zero incidence in almost-free-molecule flow

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A physical theory is proposed for the skin friction on a flat plate at zero incidence in the transition flow regime, i.e. in the flow of a moderately rarefied gas. The ratio of the molecular mean free path to the characteristic size of the plate is assumed of order unity or larger. A general formula for the perturbation to the well-known friction of the free-molecule theory is given. This perturbation is attributed to the intermolecular collisions which are neglected on the basis of the free-molecule hypothesis. The expected rate of collisions are calculated for rigid spheres, using the classical kinetic theory.

Although this is intended as an approximate theory, the theoretical results check surprisingly well with the limited experimental data that are available. The present theory shows that the ratio of the Reynolds number to the Mach number squared is the governing parameter for determining the intermolecular collision effect on skin friction in the transition flow regime.

1. Introduction

The treatment of rarefied gas dynamics has been greatly simplified by the use of the free-molecule hypothesis when the molecular mean free path λ is many times greater than the characteristic dimension L of the particular flow field. In this hypothesis it is assumed that the gas dynamic effects of collisions between the molecules incident on and reflected from a surface element are negligible compared to those between the incident molecules and the surface element. This basic postulate allows the molecular motion of the incident stream to be treated as having the Maxwellian (equilibrium) distribution. Gas dynamic problems of this nature are amenable to solution provided the mechanism of molecular reflexions at the solid surface is known.

Similarly, treatment of gas dynamic problems in the slip flow regime, where the ratio λ/L is much less than unity, has been moderately successful, at least in a few specific cases (Schaaf 1956).

On the other hand, the least rewarding efforts have been in the study of flow phenomena in the transition regime, where λ is of the same order as L. Mathematically, the classical approaches to such rarefied gas problems always start with the Maxwell–Boltzmann equation for the molecular distribution function. The general technique (Jaffé 1930; Keller 1948; Wang Chang & Uhlenbeck 1954) is based on a perturbation expansion of the Maxwell–Boltzmann equation in powers of L/λ. The calculations are in general formidable and few results of direct aerodynamic interest are available.

Unfortunately, experimental investigations in the transition flow regime have also been fraught with difficulties. From the astronautical point of view, the flow conditions existing in the transition regime correspond approximately to the case of ballistic missile flight at an altitude of about 100 km.

It appears that for the purpose of elucidating the physical mechanism of the phenomena, one might use a physical approach entailing a slight relaxation of the conditions imposed by the free-molecule hypothesis. The proposed principle of treating this 'almost-free-molecule flow' starts with the following assumptions: (1) The rate of collisions between the molecules incident on and reflected from a surface element is small compared to that between the incident molecules and the surface element in question. (2) The probability of a reflected molecule colliding twice with the incident molecules before it is deflected away is negligible compared to the probability of its colliding only once. The single-collision* effects are calculated on the basis of Maxwellian distribution for the molecular velocities. In other words, the theory of the almost-free-molecule flow is essentially a higher order iteration of the corresponding free-molecule analysis. The intermolecular collisions are calculated for rigid spheres based on the classical kinetic theory. (3) The molecules are reflected diffusely from the surface without preferred direction.

As in other mean-free-path methods of treating kinetic problems, the almost-free-molecule calculation is expected to indicate quickly the order of magnitude of certain gas dynamic quantities in question and their functional dependence upon molecular variables. It should be kept in mind, however, that this approximate theory, like many other elementary theories of complicated phenomena, can by no means be regarded as a complete equivalent of an exact theory such as the yet-to-be-obtained solution to the Maxwell–Boltzmann equation for the case of a transition flow along a plate. Nevertheless, the results predicted by the theory for both the drag of a flat plate at zero incidence and the Pitot pressure (Liu 1957) in the transition flow regime were found to show remarkable consistency with the limited measurements available.

2. Flat plate in a free-molecule flow

Consider a flat plate situated in a free stream of infinite extent with mean velocity V_1. If the intermolecular collisions occurring near the plate are negligible, we may use the Maxwellian distribution for the velocities of the impinging molecules relative to an observer moving with the stream. Assuming that the plate is in the y, z-plane such that the direction of flow makes an angle α with the y-axis and is perpendicular to the z-axis, we have the distribution function (Jeans 1925)

$$f = (\beta^3/\pi^{\frac{3}{2}})\exp\{-\beta^2[(u-V_1\sin\alpha)^2+(v+V_1\cos\alpha)^2+w^2]\}, \tag{2.1}$$

where $\beta = (2RT_1)^{-\frac{1}{2}}$ denotes the reciprocal of the most probable velocity of the molecules relative to the mean motion of the gas; T_1, the free-stream temperature; and R, the gas constant.

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* Heineman (1948) earlier used a similar idea in treating the drag on a flat blunt body in very-high-speed flow.

The number of molecules, N_i, that are incident on a unit area per unit time is given by

$$N_i = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty} n_1 uf\,du\,dv\,dw, \tag{2.2}$$

where n_1 denotes the number density of molecules in the free stream. Equation (2.2), on integration, becomes

$$N_i = n_1\left(\frac{RT_1}{2\pi}\right)^{\frac{1}{2}}\{\exp(-S_1^2\sin^2\alpha)+\sqrt{\pi}\,S_1\sin\alpha[1+\operatorname{erf}(S_1\sin\alpha)]\}, \tag{2.3}$$

where $S_1 = V_1/(2RT_1)^{\frac{1}{2}}$, and subscript 1 denotes the free-stream condition.

To calculate the shearing stress, τ_i, due to the impact of the incident molecules on the plate, we have

$$\tau_i = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty} n_1 muvf\,du\,dv\,dw, \tag{2.4}$$

where m denotes the molecular mass. For the special case where $\alpha = 0$, $\tau_i/\frac{1}{2}\rho_1 V_1^2 = 1/\sqrt{\pi}\,S_1$ (Patterson 1956), where ρ_1 denotes the free-stream density.

If the molecules are reflected diffusely from the plate, $\tau_r = 0$ (where τ_r denotes the shearing stress due to the reflected molecules), since no preferred direction exists. Thus, under the free-molecule hypothesis, the drag coefficient of a flat plate, C_{DF}, becomes

$$C_{DF} = 2\tau_i/\tfrac{1}{2}\rho_1 V_1^2 = 2/\sqrt{\pi}\,S_1. \tag{2.5}$$

3. Rate of intermolecular collisions and loss of shearing stress

It is obvious that loss of shearing stress on the plate can result when the incident molecules are prevented from reaching the plate by intermolecular collisions. On the other hand, the molecules emerging from such collisions may still strike the plate and thereby contribute additional shearing stress. From this it follows that in an almost-free-molecule flow the net loss of shearing stress (relative to its 'free-molecule' value) due to the intermolecular collisions is equal to the difference between these two aforementioned contributions.

To calculate the potential contribution to shearing stress by the incident molecules about to collide with the diffusely reflected molecules, we first need to know the rate of such intermolecular collisions for each incident molecule. Let c_i denote the absolute velocity of a molecule incident upon a plate, n_0 and c_0 denote, respectively, the number density and the mean velocities of molecules re-emitted randomly from the plate. If $\bar{C}_r$ represents the mean relative velocities of collisions between the incident molecule, with velocity c_i, and the diffused molecules, with a mean velocity $\bar{c}_0$, we have the expected collision rate (Kennard 1938)

$$E_0 = \pi\sigma^2 n_0 \bar{C}_r, \tag{3.1}$$

where $\pi\sigma^2$ denotes the collision cross-section of the equivalent rigid spheres. Since

$$\bar{C}_r = \int_{\theta_i}^{\theta_i+\frac{1}{2}\pi} \tfrac{1}{2}C_r\sin\theta\,d\theta, \tag{3.2}$$

where $\theta = \theta_i + \theta_0$ (see figure 1), and

$$C_r^2 = c_i^2 + \bar{c}_0^2 + 2\bar{c}_0 c_i \cos\theta, \tag{3.3}$$

we have from (3.1)

$$E_0 = -\frac{\pi\sigma^2 n_0}{6\bar{c}_0 c_i}(c_i+\bar{c}_0)^3 \left\{ \left[1 - \frac{2\bar{c}_0 c_i}{(c_i+\bar{c}_0)^2}(1+\sin\theta_i)\right]^{\frac{3}{2}} - \left[1 - \frac{2\bar{c}_0 c_i}{(c_i+\bar{c}_0)^2}(1-\cos\theta_i)\right]^{\frac{3}{2}} \right\}. \tag{3.4}$$

Note that the term $2(\bar{c}_0/c_i)/(1+\bar{c}_0/c_i)^2$ is always small compared to unity. We can obtain a fair approximation by retaining the first-order terms only in the power series expansion of E_0 in equation (3.4). The approximate rate of collisions at the plate becomes

$$E_0 = \tfrac{1}{2}\pi\sigma^2 n_0 (c_i + \bar{c}_0)(\sin\theta_i + \cos\theta_i). \tag{3.5}$$

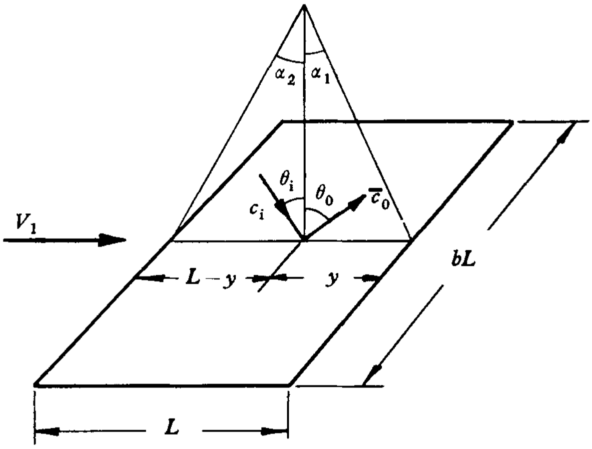

FIGURE 1. Velocity vectors of the molecular motion of a gas flowing along a flat plate.

At distance x from the plate, the collision rate is reduced by the following factors: (1) the molecular scattering effect, $\exp(-x/\lambda_1)$; (2) the bilateral free expansion of the reflected molecules, $bL^2/(x+L)(x+bL)$, where b denotes the aspect ratio of the span to the chord of the plate (see figure 1). Therefore, the likelihood of collisions by an incident molecule between distances x and $x+dx$ becomes

$$E\,dt = E\frac{dx}{u} = \frac{\frac{1}{2}bL^2\pi\sigma^2 n_0}{(x+L)(x+bL)} e^{-x/\lambda_1}(c_i+\bar{c}_0)(\sin\theta_i+\cos\theta_i)\frac{dx}{u}. \tag{3.6}$$

If all the collided molecules are deflected away from the plate, we can calculate the potential loss of shearing stress with the use of equations (2.4) and (3.6). To avoid mathematical complexity, we can get a fair approximation by using the mean incident velocity [equation (2.3)] at a given angle θ_i without directly taking into account the random distribution of the molecular velocities. To carry the approximation further, we consider only those intermolecular collisions that occur in the semi-infinite region based on the plate.

If there were no collisions, the tangential momentum integral of the incident molecules that would have hit the plate can be written as

$$\tau_{11} = \frac{b}{2}\pi\sigma^2 n\rho_1 V_1^2 L \int_0^\infty \frac{(c_i/V_1)(\bar{c}_0+c_i/V_1)}{(x+L)(x+bL)} e^{-x/\lambda_1} \int_0^L \int_{\alpha_1}^{-\alpha_2} \sin\theta_i(\sin\theta_i+\cos\theta_i)\,d\theta_i\,dy\,dx, \tag{3.7}$$

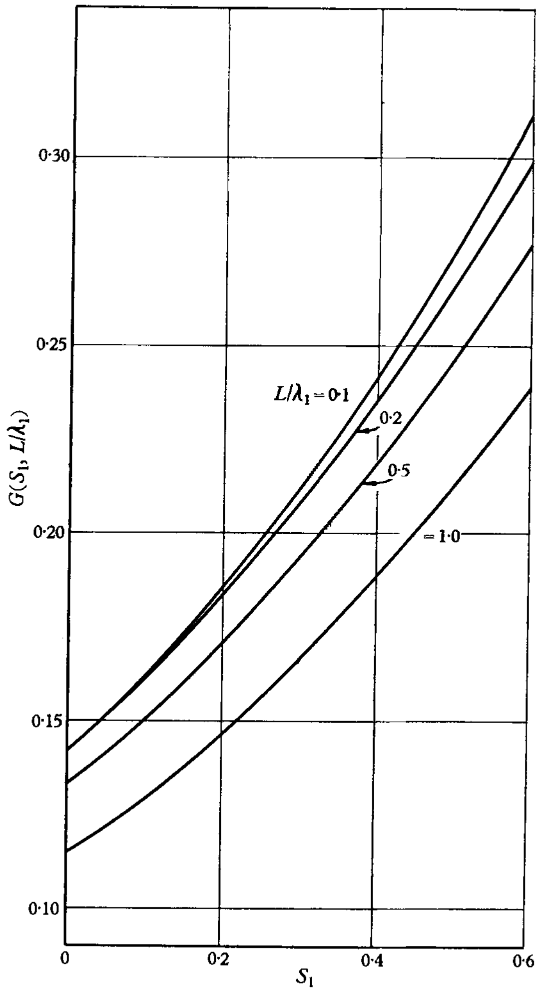

FIGURE 2. G-function for a plate of infinite span ($b = \infty$).

where $\alpha_1 = \tan^{-1}(y/x)$, $\alpha_2 = \tan^{-1}[(L-y)/x]$, and c_i is taken to be the mean velocity

$$\bar{c}_i = \left(\frac{RT_1}{2\pi}\right)^{\frac{1}{2}} \{\exp(-S_1^2\sin^2\theta_i)+\sqrt{\pi}\,S_1\sin\theta_i[1+\operatorname{erf}(S_1\sin\theta_i)]\}. \tag{3.8}$$

For the case where the speed ratio S_1 is smaller than unity, we may approximate $\bar{c}_i$ given in equation (3.8) by the power series

$$\bar{c}_i = \left(\frac{RT_1}{2\pi}\right)^{\frac{1}{2}} [1+\sqrt{\pi}\,S_1\sin\theta_i - 3S_1^2\sin^2\theta_i + \ldots]. \tag{3.9}$$

With the approximate expression for $\bar{c}_i$, as given in equation (3.9), the multiple integrations of equation (3.7) can be performed. Thus,

$$\tau_{11} = -\tfrac{1}{2}\pi\sigma^2 n_0 \rho_1 V_1^2 (L/S_1^2)\, G, \tag{3.10}$$

where G is a function of S_1, b and L/λ_1 (see figures 2 and 3).

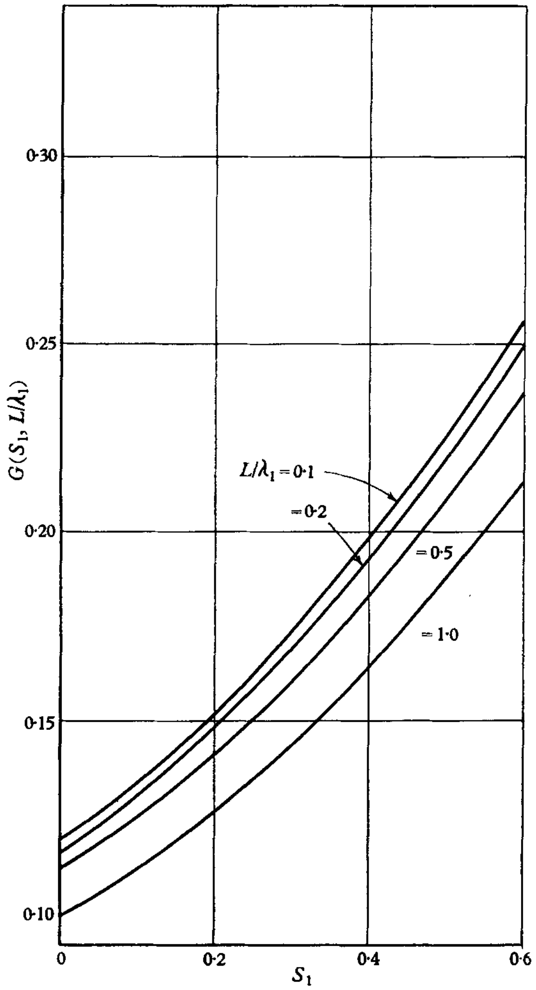

FIGURE 3. G-function for a square plate ($b = 1$).

(i) For the case $b = \infty$,

$$\begin{aligned} G(S_1, L/\lambda_1) = {} & \frac{5}{4\pi}\int_0^\infty \frac{e^{-(L/\lambda_1)t}}{1+t}\left[\cot^{-1}t - t\log\frac{1+t^2}{t^2}\right]dt + \frac{S_1}{\sqrt{\pi}}\int_0^\infty \frac{e^{-(L/\lambda_1)t}}{1+t} \\ & \times\left[(1+t^2)^{\frac{1}{2}} + \frac{t^2}{(1+t^2)^{\frac{1}{2}}} - 2t\right]dt - \frac{1}{8}\left(\frac{1}{2}+\frac{3}{\pi}\right)S_1^2\int_0^\infty \frac{e^{-(L/\lambda_1)t}}{1+t} \\ & \times\left[4t\log\frac{1+t^2}{t^2} - 3\cot^{-1}t - \frac{t}{1+t^2}\right]dt + \frac{S_1^3}{5\sqrt{\pi}}\int_0^\infty \frac{e^{-(L/\lambda_1)t}}{1+t} \\ & \times\left[(1+t^2)^{\frac{1}{2}} + 2\frac{t^2}{(1+t^2)^{\frac{1}{2}}} - \frac{1}{3}\frac{t^4}{(1+t^2)^{\frac{3}{2}}} - \tfrac{8}{3}t\right]dt. \end{aligned} \tag{3.11}$$

(ii) For the case $b = 1$,

$$\begin{aligned} G(S_1, L/\lambda_1) = {} & \frac{5}{4\pi}\int_0^\infty \frac{e^{-(L/\lambda_1)t}}{(1+t)^2}\left[\cot^{-1}t - t\log\frac{1+t^2}{t^2}\right]dt + \frac{S_1}{\sqrt{\pi}}\int_0^\infty \frac{e^{-(L/\lambda_1)t}}{(1+t)^2} \\ & \times\left[(1+t^2)^{\frac{1}{2}} + \frac{t^2}{(1+t^2)^{\frac{1}{2}}} - 2t\right]dt - \frac{1}{8}\left(\frac{1}{2}+\frac{3}{\pi}\right)S_1^2\int_0^\infty \frac{e^{-(L/\lambda_1)t}}{(1+t)^2} \\ & \times\left[4t\log\frac{1+t^2}{t^2} - 3\cot^{-1}t - \frac{t}{1+t^2}\right]dt + \frac{S_1^3}{5\sqrt{\pi}}\int_0^\infty \frac{e^{-(L/\lambda_1)t}}{(1+t)^2} \\ & \times\left[(1+t^2)^{\frac{1}{2}} + 2\,\frac{t^2}{(1+t^2)^{\frac{1}{2}}} - \frac{1}{3}\,\frac{t^4}{(1+t^2)^{\frac{3}{2}}} - \tfrac{8}{3}t\right]dt. \end{aligned} \tag{3.12}$$

4. Contribution to the shearing stress by the scattered molecules

A specific difficulty of the present problem is the determination of the shearing stress, τ_{10}, on the plate due to the impact of the molecules emerging from the intermolecular collisions. Before entering into this determination, we wish to emphasize that these evaluations are mere approximations.

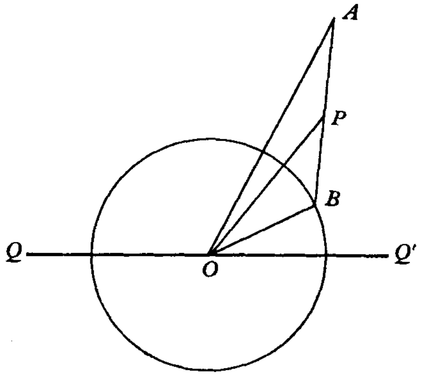

FIGURE 4. Velocity vector diagram of molecular encounters between an incident molecule and the molecules reflected from a plate.

We begin with a theorem of Maxwell (Jeans 1925). If two molecules collide with such velocities that their centre of gravity is stationary before collision and, therefore, also after collision, then all directions are equally probable for the velocity of either molecule following collision. From this it follows that, if two molecules collide in any way, the expected velocity of either is exactly equal to the velocity of the centre of gravity of the colliding pair.

Consider an incident molecule A with velocity c_i which is represented by OA in figure 4. Let $\bar{c}_0$, represented by OB, be the velocity of a reflected molecule B from the plate. Their relative velocity is therefore represented by AB; the velocity of the centre of gravity, by OP, where P is the mid-point of AB. The expected velocity of A, emerging from collision with B, is represented by OP.

In the present analysis of skin friction, we are concerned primarily, though not exclusively, with the component of OP along the plate QQ' since the tangential momentum transfer to the plate is of particular interest. By reason of symmetry in molecular distributions above and below the plate, we can determine the

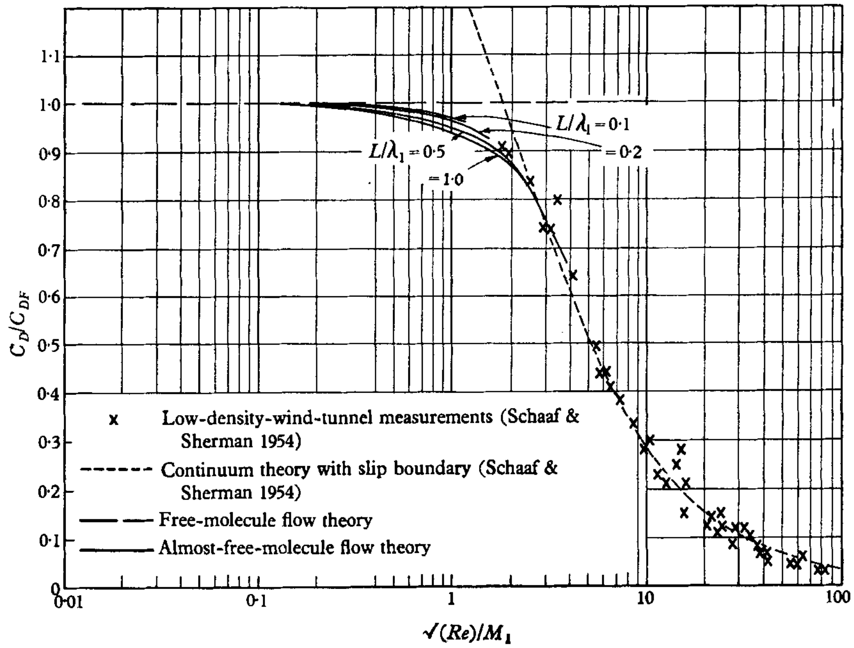

FIGURE 5. Comparison of theoretical and measured values of C_D/C_{DF} ($b = \infty$ is assumed in the almost-free-molecule-flow analysis).

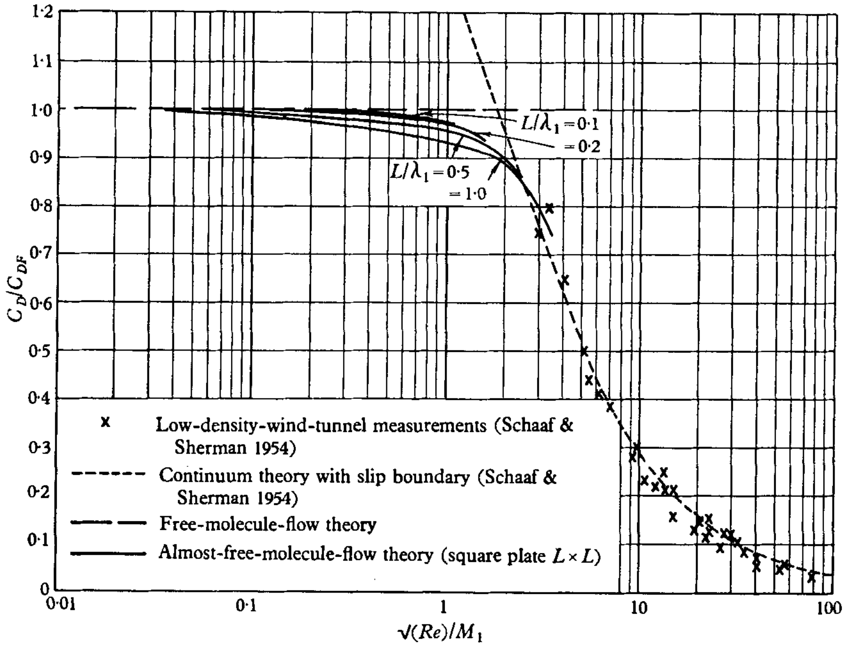

FIGURE 6. Comparison of theoretical and measured values of C_D/C_{DF} ($b = 1$ is assumed in the almost-free-molecule-flow analysis).

expectation of the final velocity of A by averaging the components of the velocity OP over all possible directions for OB. Now the similarity can be established between the analysis herein for the expected final velocity of A and the classical calculation for the *persistence of molecular velocities* in the kinetic theory of gases (Jeans 1925). After the collision with another randomly moving molecule of equal size, the velocity of a given molecule will, on the average, still retain a component in the direction of its original motion. This characteristic is called the persistence of velocities after collision. It has been shown by Jeans (1925) approximately, and by Chapman & Cowling (1952) exactly, that the mean persistence, for identical molecules in collisions, is equal to 0·406 times the original velocity.

Provided that the above-mentioned hypothesis is valid, the mean expected velocity of the centre of gravity of molecules A and B will be equal to $0{\cdot}406c_i$ in the direction OA. From this it follows that the additional shearing stress, τ_{10}, to the plate due to the impact of the molecules (both A and B) emerging from the intermolecular collisions will amount to approximately 81·2 % of τ_{11} (see equation 3.10).

5. The drag coefficient of a flat plate

Given the intermolecular collision effect on skin friction of a surface as worked out in §§ 3 and 4, the drag coefficient, C_D, of a flat plate at zero incidence in an almost-free-molecule flow follows from (2.5). Thus

$$C_D = (2/\sqrt{\pi}\, S_1) - 0{\cdot}376\pi\sigma^2 n_0 (L/S_1^2)\, G. \tag{5.1}$$

Dividing equation (5.1) by C_{DF} leads to

$$C_D/C_{DF} = 1 - 0{\cdot}188\pi^{\frac{3}{2}}\sigma^2 n_0 (L/S_1^2)\, G. \tag{5.2}$$

In the case when $T_0 = T_1$ and, therefore, $n_0 = n_1$, and since

$$\sqrt{2}\,\pi\sigma^2 n_1 = 1/\lambda, \quad S_1 = (\gamma/2)^{\frac{1}{2}} M_1 \quad \text{and} \quad L/\lambda_1 = 0{\cdot}499(8/\pi\gamma)^{\frac{1}{2}}\, Re/M_1,$$

where M_1 and Re denotes the free-stream Mach number and Reynolds number respectively, we have

$$C_D/C_{DF} = 1 - 0{\cdot}188(Re/M_1^2)\, G. \tag{5.3}$$

The comparison of C_D/C_{DF} computed from equation (5.3), for a two-dimensional plate ($b = \infty$) and a square plate ($b = 1$) at various values of S_1 and L/λ_1, with those obtained from measurements in a low-density wind tunnel (Schaaf & Sherman 1954) are shown in figures 5 and 6, respectively.

6. Conclusions

Considering the number of serious simplifications made in the formulation process, we cannot expect the theory to predict results with greater accuracy than the right order of magnitude. The surprisingly close agreement between the theory and the limited measurements that are available should be taken with guarded optimism until a wider variety of experimental results is available.

The present theory does show that Re/M^2 is the governing parameter for the determination of the intermolecular collision effect on skin friction of a shear flow in the transition regime. It is of interest to compare the present result on intermolecular collision effect with that on Pitot pressure in an almost-free molecule flow (Liu 1957). In the case of Pitot pressure, Re/M_1 instead of Re/M_1^2 becomes the corresponding governing parameter. This exhibits the difference in physical mechanisms between the shear flow effect (skin friction) and the impact flow effect (Pitot pressure) in the transition regime.

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REFERENCES

CHAPMAN, S. & COWLING, T. G. 1952 *Mathematical Theory of Non-Uniform Gases.* Cambridge University Press.

HEINEMAN, M. 1948 *Commun. Appl. Math.* **1**, 259.

JAFFÉ, G. 1930 *Ann. Physik*, **6**, 195.

JEANS, J. H. 1925 *Dynamical Theory of Gases.* Cambridge University Press.

KELLER, J. B. 1948 *Commun. Appl. Math.* **1**, 275.

KENNARD, E. H. 1938 *Kinetic Theory of Gases.* New York: McGraw-Hill.

LIU, V. C. 1957 *On Pitot Pressure in an Almost-Free-Molecule Flow—a physical Theory of Flow of the Rarefied Gases.* High Altitude Engineering Laboratory Research Paper no. 18, The University of Michigan, Ann Arbor; also *J. Aero. Sci.* **25**, 1958, 779.

PATTERSON, G. N. 1956 *Molecular Flow of Gases.* New York: Wiley.

SCHAAF, S. A. 1956 *Appl. Mech. Rev.* **9**, 413.

SCHAFF, S. A. & SHERMAN, F. S. 1954 *J. Aero. Sci.* **21**, 85.

WANG CHANG, C. S. & UHLENBECK, G. E. 1954 *The Couette Flow between two Parallel Plates as a Function of the Knudsen Number.* The University of Michigan Engineering Research Institute, Project 1999.